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Elastodynamics of Cracked Structures Using Finite Elements

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WITHIN the past few years, a number of special crack-tip finite elements have been developed[1–6] for application to cracked structural components whose irregular geometry or otherwise inconvenient boundary conditions dictate a numerical analysis. Because these elements include in their formulation the $r^{-1/2}$ crack-tip stress singularity inherent to two-dimensional fracture mechanics, their incorporation in a finite-element model permits an accurate and economical assessment of the stresses near a crack tip. Specifically and most importantly, the singularity element gives directly the value of the plane-deformation stress-intensity factors K_I and K_{II} for the particular loads and crack length under consideration. All applications of this type with which the authors are familiar have been either for equilibrium problems or problems for which the inertia forces can reasonably be neglected. In this Note we present applications in elastodynamics of two singularity elements whose stiffnesses were characterized earlier by Aberson, Anderson, and Hardy.[7–8] The elements, one for problems which are symmetric with respect to the crack (so that only the crack-opening mode of deformation is operative) and one for general plane applications (involving both the opening and sliding modes), are shown in Fig. 1.

For symmetric problems only half of the region near the tip of a crack need be represented. The singularity element used for such applications is shown in Fig. 1a. It is rectangular with a 3:1 aspect ratio and has 8 nodes. The 16 displacement degrees of freedom correspond to the first 13 symmetric modes of deformation as given by Williams[9] plus the 3 rigid-body degrees of freedom. The crack tip is located midway along the bottom edge of the element at the origin of the polar coordinate system. The upper crack face extends from this point horizontally to the left. As dictated by symmetry, nodes along the prolongation of the crack are restrained against vertical displacement. For asymmetric problems the entire region in the immediate vicinity of the crack

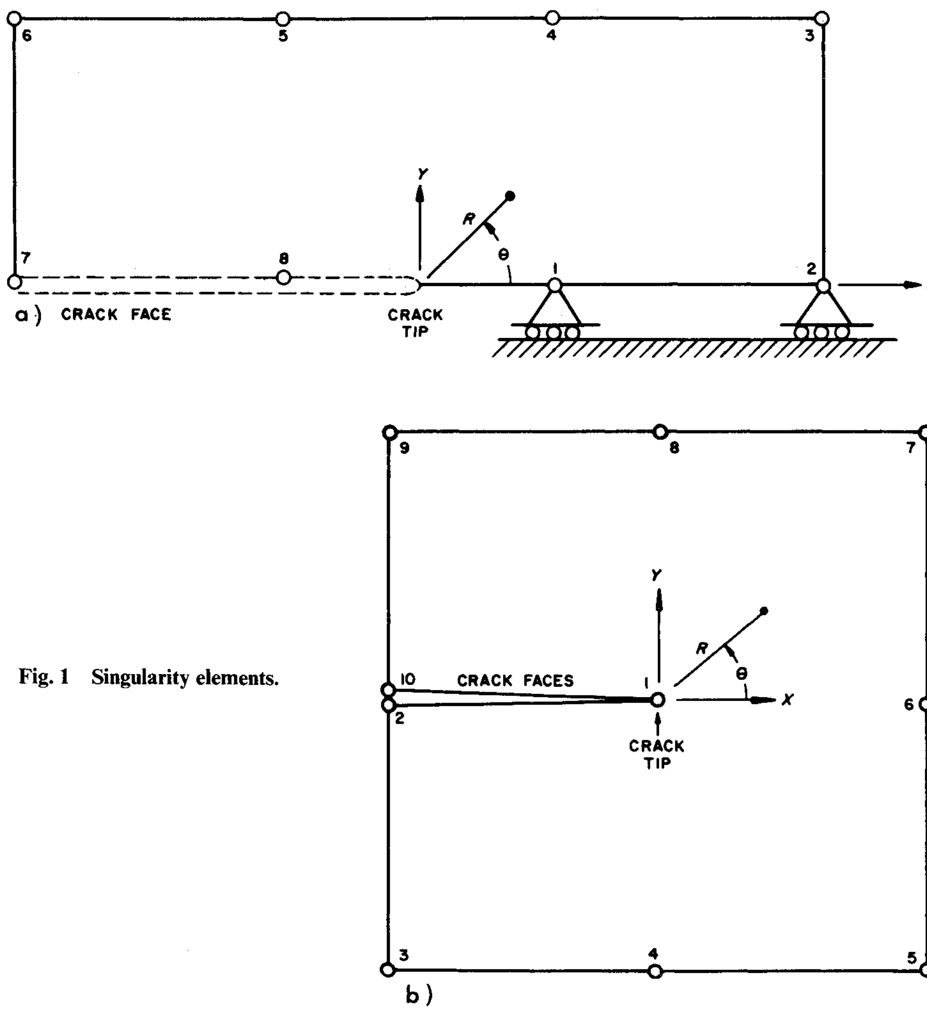

Fig. 1 Singularity elements.

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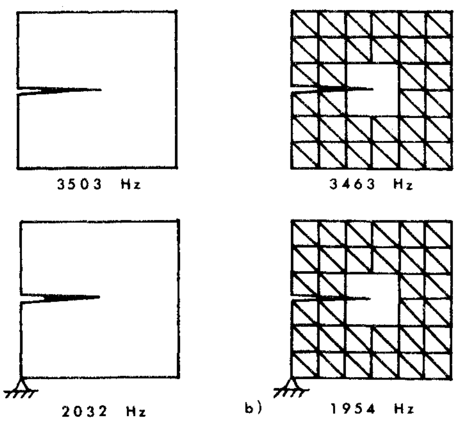

Fig. 2 Fundamental frequencies of 6 in. × 6 in. × 0.5 in. aluminum plate.

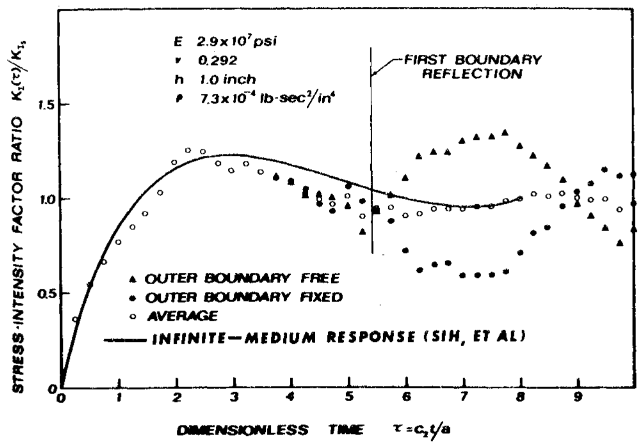

Fig. 4 Time dependence of stress-intensity factor.

tip is represented by the square singularity element shown in Fig. 1b. The element's 10 nodes require 20 degrees of freedom for independence of the nodal-displacement components. In addition to the 3 rigid-body terms, these are taken to be the first 9 symmetric and first 8 antisymmetric Williams' modes. Both of the singularity elements shown in Fig. 1 have shapes and nodal positions which make them convenient to use with constant-strain triangles. The displacement incompatibility that exists where a singularity element interfaces with a constant-strain triangle appears to be inconsequential in light of the exceptional accuracy which has been economically achieved in static applications.[7]

For problems of elastodynamics the element must be assigned inertia properties. We have chosen the "consistent mass" characterization. Calculation of the mass-matrix elements requires numerical integration over the element volume. Elements of the mass matrix were determined to 4-figure accuracy; in the case of the 10-node element 4096 integration points were employed. This expensive quadrature must be performed only once for each of these homogeneous, constant-thickness elements. This is possible because the particular form of the near-tip displacement components permits decomposition of each mass matrix into 3 matrices, which, except for their coefficients, are independent of the element size and material properties. A similar decomposition was used in the computation and storage of the stiffness matrices.

The singularity elements were developed for application to plane cracked bodies of finite dimension. Analytical solutions for this problem class are practically nonexistent. Consequently, even with the elements' inertia properties tabulated, an assessment of kinetic performance by direct comparison with existing solutions was not possible. An alternate measure of adequacy in a dynamic situation is representation of natural frequencies. A plate reminiscent of a compact test specimen (Fig. 2a) was modeled by the 10-node element alone. Material parameters were chosen to correspond nominally to aluminum. The first nonzero natural frequencies were computed for the plate free of constraint and for one corner restrained against displacement normal to the crack. The computations were repeated for a second finite-element representation (Fig. 2b) in which the singularity element, representing only the central $\frac{1}{9}$ of the plate, was surrounded by constant-strain triangles. The natural frequencies computed using the coarse model were found to exceed by less than 5% those for the more refined representation. The refined model had the virtues of a larger number of degrees of freedom and relative insensitivity to the characteristics of the singularity element. The accuracy of the natural frequencies predicted by the singularity element alone implies its utility in relatively coarse-grid applications.

Figure 3 shows a problem solved by Sih, Embley, and Ravera[10] and the finite-element model for that problem. The crack of length $2a$ in an initially quiet infinite two-dimensional solid is face loaded at time $t = 0$ by uniform pressure of magnitude p. The dimensions of the finite-element model are indicated in Fig. 3; due to symmetry only one quadrant was modeled. The material is steel with a wave speed ratio $C_2/C_1 = 0.542$. The Newmark-β method of time integration (with $\beta = \frac{1}{4}$) was used. Results are presented in Fig. 4, where the ratio of dynamic stress-intensity factor $K_I(\tau)$ to static stress-intensity factor $K_{I_s} = p(\pi a)^{1/2}$ is plotted against nondimensional time $\tau = C_2 t/a$, where C_2 is the shear-wave speed. To assess the effects of the outer boundary, the finite-element problem was run with the remote-boundary nodes fixed and with them free. The usual discrete-model effect of "smearing forward" a wavefront is apparent from Fig. 4. Elastodynamics predicts that the presence of the outer boundary cannot be felt at the crack tip prior to about $\tau = 5.5$, whereas the finite-element results indicate an effect at about $\tau = 4$. The arithmetic average of the boundary-fixed and the boundary-free responses closely approximates the infinite-medium response, which is reasonable, since the far-field motion should be nearly one-dimensional for a crack half-length small in comparison with the plate radius. Solution times for Sih's problem with 400 time steps, including mass and stiffness matrix generation and assembly, were slightly less than 130 sec on a UNIVAC 1108.

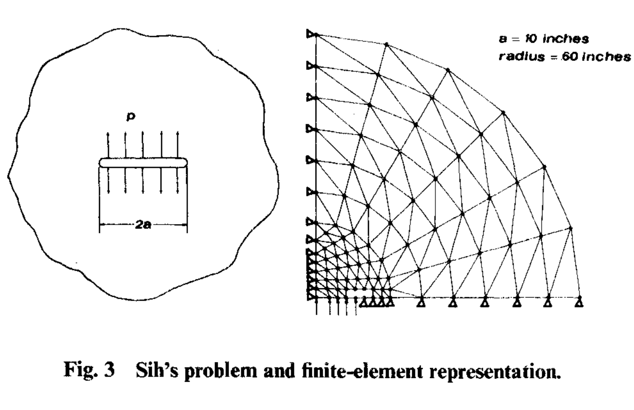

Fig. 3 Sih's problem and finite-element representation.

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Numerical Solution of a Nonlinear Integral Equation by the Use of Step Functions: III. Continuous Solutions

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Nomenclature

A = matrix defined in Eq. (5)
B = integrated quantity of Planck's function, $\theta^4 I(z)$
E_n = exponential integral of order n
F = nondimensional boundary flux, or function of variable Ψ
f = boundary function defined in Eq. (3)
g = boundary function
h = Planck's constant
I = one-half the total step number, or

$$I(z) = 15/\pi^4 \int_0^\infty \delta(\alpha_v) z^3\, dz/(e^{-z}-1)$$

J = functional
K = kernel function
Ki = indefinite integral of the kernel function
k = Boltzmann's constant
M = number of iterations
N = number of spectral band

$$\operatorname{sgn}(x) = \begin{cases} + & \text{if } x > 0 \\ - & \text{if } x < 0 \end{cases}$$

T = temperature, or integral of the kernel function, Eq. (9)
z = variable, $\bar{v}/\theta$
α = absorption coefficient
δ = delta function, $\delta(\alpha_v) = \begin{cases} 1 & \text{for frequencies where } \alpha_v = \alpha_k \\ 0 & \text{otherwise} \end{cases}$
Δ = constant increment in step function approximating the function Ψ
Δv_k = increment of frequency, $v_{k+1} - v_k$
η = optical thickness, $-l \leq \eta \leq l$
$\bar{\eta}$ = dimensionless optical thickness, η/l
θ = normalized temperature, T/T_B
l = one-half the optical thickness
λ = constant in integral equation
v = frequency
$\bar{v}$ = nondimensional frequency, hv/kT_B
σ = Stefan-Boltzmann constant
ϕ = variational function
Ψ = solution
ψ = step solution
τ_w = total optical thickness, $2l$

Subscripts

A = at one boundary
B = at the other boundary
I = at Ith point
i, j = indexes of summation, or at ith or jth step point
k = frequency interval corresponding to kth spectral band
v = frequency

Superscript

$-$ = normalized

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Introduction

INTEGRAL equations play an important role in the development of many problems in physics and fluid dynamics, e.g., in the boundary value problems involved in kinetic theory, in the radiative transfer equation, in wing theory, in Boltzmann's equation, and in Schrodinger's equation for the quantum theory of scattering. For problems that require numerical solutions involving complicated sets of boundary conditions, it may be more practical to obtain solutions rapidly with moderate accuracy than time consuming exact solutions. Such cases arise in the calculation of the transition probabilities for use in the Monte Carlo simulation of reacting flow and in the classical master equations for the nonequilibrium dissociation and recombination reactions, for example.

In previous studies,[1-3] the author has demonstrated the advantages of the step function incorporated with the variational technique. It is, however, often desirable to have a continuous solution, rather than the step function solution, especially near the boundary point where the values of solution change rapidly. This Note will demonstrate that accurate continuous solutions can be derived from the results of the previous step function solutions.

Analysis

Basic Equation

The basic nonlinear integral equation considered in the present problem may be written in symbolic notation as

$$F(\Psi) = g + \lambda K \Psi \tag{1}$$

where F is a given function of the unknown variable Ψ, g is the boundary function, λ is a given constant, and K is the symmetric singular kernel. (Here, we simply assume a unique solution exists.)

For the purpose of numerical calculation, Eq. (1) is recast into the standard form of Fredholm's equation (second kind)

$$\Psi = f + \lambda K \Psi \tag{2}$$

where

$$f = \Psi - F(\Psi) + g \tag{3}$$

The f function in Eq. (2) is treated as a known quantity since, in practice, an (arbitrary) initial guess for Ψ can be used.

A solution to the linear integral Eq. (2) is obtained by the variational method, which extremizes the functional J

$$J(\phi) = 1/2\,\|\phi\|^2 - 1/2(\lambda K\phi + 2f)\phi \tag{4}$$

The functional J takes a minimum value at $\phi = \Psi$, which is the solution of Eq. (2).

Step Function Solution

The function Ψ is approximated stepwise by a number I of constants

$$\Delta_i,\ \psi_i = \sum_{j=1}^{i} \Delta_j$$